An upper triangular form of the tridiagonal matrix may be obtained as follows:

$$b_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1} \quad i = 2, 3, \ldots NI$$

$$g_i = g_i - \frac{a_i}{b_{i-1}} g_{i-1} \quad i = 2, 3, \ldots NI$$

$$T_{NI} = \frac{g_{NI}}{b_{NI}}$$

$$T_j = \frac{g_j - c_j T_{j+1}}{b_j} \quad j = NI - 1, \quad NI - 2, \ldots, 1$$

It should be noted that Neumann boundary conditions can also be accommodated into this algorithm with the tridiagonal form still maintained.

4.3 HYPERBOLIC EQUATIONS

Hyperbolic equations, in general, represent wave propagation. They are given by either first order or second order differential equations, which may be approximated in either explicit or implicit forms of finite difference equations. Various computational schemes are examined below.

4.3.1 EXPLICIT SCHEMES AND VON NEUMANN STABILITY ANALYSIS

Euler's Forward Time and Forward Space (FTFS) Approximations

Consider the first order wave equation (Euler equation) of the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a > 0 \tag{4.3.1}$$

The Euler's forward time and forward space approximation of (4.3.1) is written in the FTFS scheme as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x} \tag{4.3.2}$$

It follows from (4.2.15) and (4.3.2) that the amplification factor assumes the form

$$g = 1 - C(e^{I\phi} - 1) = 1 - C(\cos\phi - 1) - IC\sin\phi = 1 + 2C\sin^2\frac{\phi}{2} - IC\sin\phi \tag{4.3.3}$$

with C being the Courant number or CFL number [Courant, Friedrichs, and Lewy, 1967],

$$C = \frac{a\Delta t}{\Delta x}$$

and

$$|g|^2 = g\,g^* = \left(1 + 2C\sin^2\frac{\phi}{2}\right)^2 + C^2\sin^2\phi = 1 + 4C(1 - C)\sin^2\frac{\phi}{2} \geq 1 \tag{4.3.4}$$

where g^* is the complex conjugate of g. Note that the criterion $|g| \leq 1$ for all values of ϕ can not be satisfied ($|g|$ lies outside the unit circle for all values of ϕ, Figure 4.3.1). Therefore, the explicit Euler scheme with FTFS is unconditionally unstable.

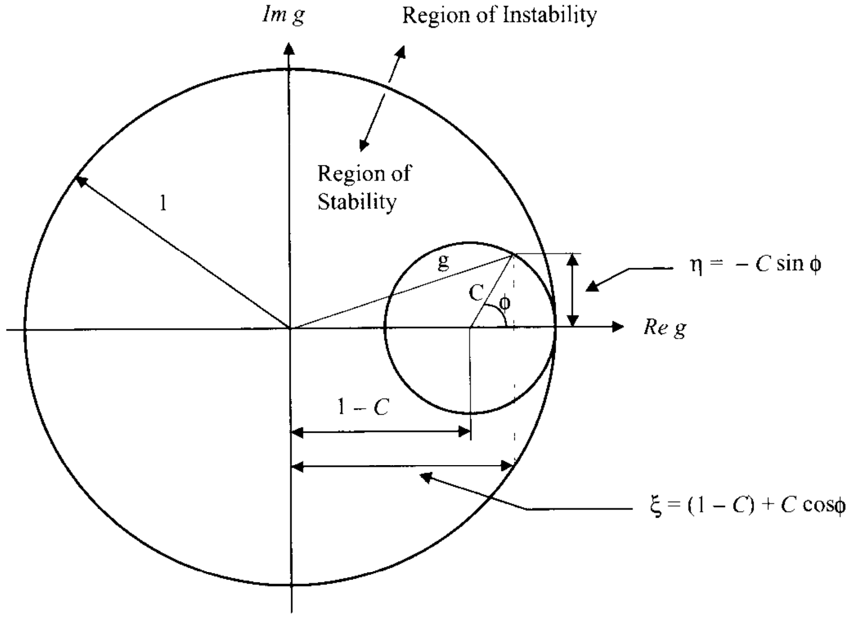

Figure 4.3.1 Complex g plane for upwind scheme with unit circle representing the stability region.

Euler's Forward Time and Central Space (FTCS) Approximations

In this method, Euler's forward time and central space approximation of (4.3.1) is used:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -a\frac{\left(u_{i+1}^n - u_{i-1}^n\right)}{2\Delta x}, \quad O(\Delta t, \Delta x) \tag{4.3.5}$$

The von Neumann analysis shows that this is also unconditionally unstable.

Euler's Forward Time and Backward Space (FTBS) Approximations – First Order Upwind Scheme

The Euler's forward time and backward space approximations (also known as upwind method) is given by

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}, \quad O(\Delta t, \Delta x) \tag{4.3.6}$$

The amplification factor takes the form

$$\begin{aligned} g &= 1 - C(1-e^{-I\phi}) = 1 - C(1-\cos\phi) - I\sin\phi \\ &= 1 - 2C\sin^2\frac{\phi}{2} - IC\sin\phi \end{aligned} \tag{4.3.7}$$

or

$$g = \xi + I\eta, \qquad |g| = \left[1 - 4C(1-C)\sin^2\frac{\phi}{2}\right]^{1/2} \tag{4.3.8a,b}$$

with

$$\begin{aligned} \xi &= 1 - 2C\sin^2\frac{\phi}{2} = (1-C) + C\cos\phi \\ \eta &= -C\sin\phi \end{aligned}$$

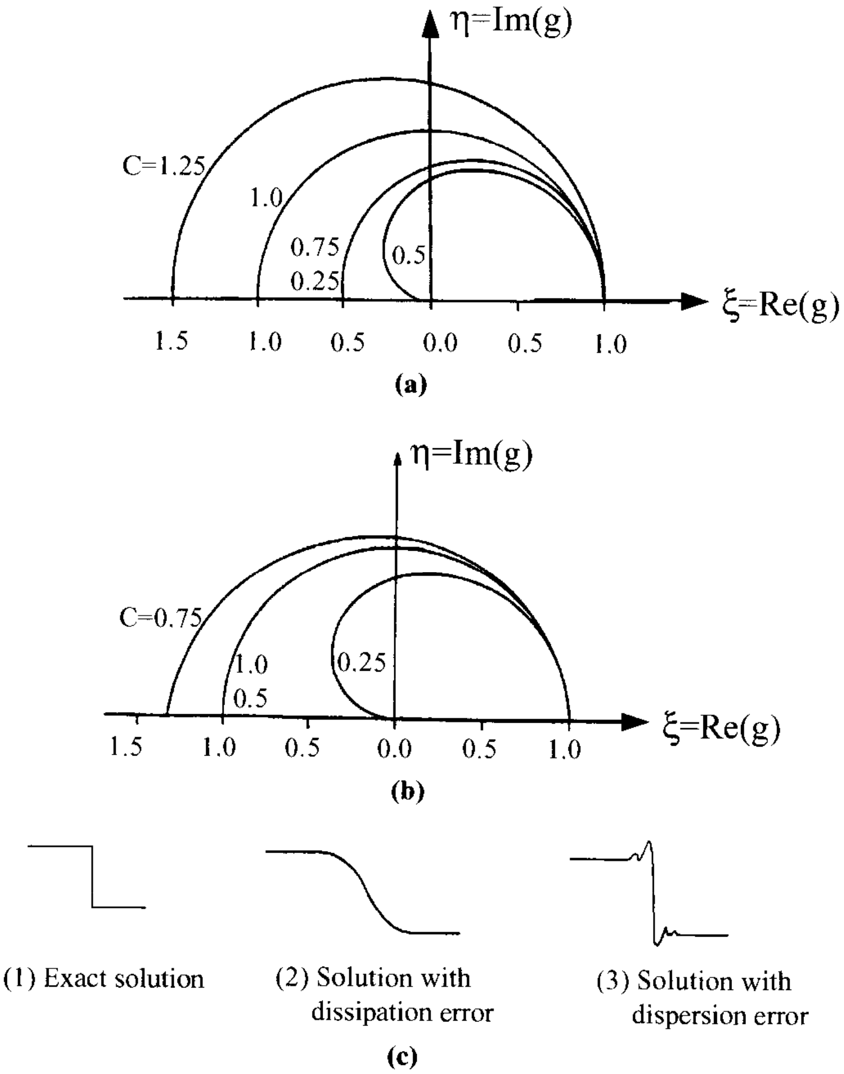

Figure 4.3.2 Dissipation and dispersion errors compared to exact solution. **(a)** Dissipation error (amplification factor modulus $|g|$). **(b)** Dispersion error (relative phase error, $\Phi/\tilde{\Phi}$). **(c)** Comparison of exact solution with dissipation error and dispersion error for shock tube problem.

which represents the parametric equation of a unit circle centered on the real axis ξ at $(1 - C)$ with radius C (Figure 4.3.1), whereas the modulus of the amplification factor, $|g|$, for various values of C are shown in Figure 4.3.2a.

In this complex plane of g, the stability condition (4.3.7) states that the curve representing g for all values of $\phi = k\Delta x$ should remain within the unit circle. It is seen that the scheme is stable for

$$0 < g < 1 \tag{4.3.9}$$

Hence, the scheme (4.3.6) is conditionally stable. Equation (4.3.9) is known as the Courant-Friedrich-Lewy (CFL) condition.

We have so far discussed the amplification factor g which represents dissipation error (Figure 4.3.2a). In numerical solutions of finite difference equations, we are also concerned with dispersion (phase) error as shown in Figure 4.3.2b. The phase Φ as determined by the adopted numerical scheme is given by the arctangent of the ratio of imaginary and real parts of g,

$$\Phi = \tan^{-1}\frac{\text{Im}(g)}{\text{Re}(g)} = \tan^{-1}\frac{\eta}{\xi} = \tan^{-1}\frac{-C\sin\phi}{1 - C + C\cos\phi} \tag{4.3.10}$$

The phase angle $\tilde{\Phi}$ is

$$\tilde{\Phi} = ka\Delta t = C\phi \tag{4.3.11}$$

The dispersion error or relative phase error is defined as

$$\varepsilon_\phi = \frac{\Phi}{\tilde{\Phi}} = \frac{\tan^{-1}[(-C\sin\phi)/(1-C+C\cos\phi)]}{C\phi} \tag{4.3.12a}$$

or

$$\varepsilon_\phi \approx 1 - \frac{1}{6}(2C^2 - 3C + 1)\,\phi^2 \tag{4.3.12b}$$

As shown in Figure 4.3.2b, the dispersion error is said to be "leading" for $\varepsilon_\phi > 1$.

The dissipation error and dispersion error for a shock tube problem can be compared to the exact solution. This is demonstrated in Figure 4.3.2c. Here, we must choose computational schemes such that dissipation and dispersion errors are as small as possible. To this end, we review the following well-known methods.

Lax Method

In this method, an average value of u_i^n in the Euler's FTCS is used:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{C}{2}\left(u_{i+1}^n - u_{i-1}^n\right) \tag{4.3.13}$$

The von Neumann stability analysis shows that this scheme is stable for $C \leq 1$.

Midpoint Leapfrog Method

Central differences for both time and spaces are used in this method:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\frac{a\left(u_{i+1}^n - u_{i-1}^n\right)}{2\Delta x}, \quad O(\Delta t^2, \Delta x^2) \tag{4.3.14}$$

This scheme is stable for $C \leq 1$. It has a second order accuracy, but requires two sets of initial values when the starter solution can provide only one set of initial data. This may lead to two independent solutions which are inaccurate.

Lax-Wendroff Method

In this method, we utilize the finite difference equation derived from Taylor series,

$$u(x, t+\Delta t) = u(x,t) + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3) \tag{4.3.15a}$$

or

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3) \tag{4.3.15b}$$

Differentiating (4.3.1) with respect to time yields

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = a^2\frac{\partial^2 u}{\partial x^2} \tag{4.3.16}$$

Substituting (4.3.1) and (4.3.16) into (4.3.15b) leads to

$$u_i^{n+1} = u_i^n + \Delta t\left(-a\frac{\partial u}{\partial x}\right) + \frac{\Delta t^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2}\right) \tag{4.3.17}$$

Using central differencing of the second order for the spatial derivative, we obtain

$$u_i^{n+1} = u_i^n - a\,\Delta t\left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right) + \frac{1}{2}(a\,\Delta t)^2\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.18}$$

This method is stable for $C \leq 1$.

4.3.2 IMPLICIT SCHEMES

Implicit schemes for approximating (4.3.1) are unconditionally stable. Two representative implicit schemes are Euler's FTCS method and the Crank-Nicolson method.

Euler's FTCS Method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{-a}{2\Delta x}(u_{i+1}^{n+1} - u_{i-1}^{n+1}), \quad O(\Delta t, \Delta x^2) \tag{4.3.19}$$

or

$$\frac{C}{2}u_{i-1}^{n+1} - u_i^{n+1} - \frac{C}{2}u_{i+1}^{n+1} = -u_i^n \tag{4.3.20}$$

Crank-Nicolson Method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right], \quad O(\Delta t^2, \Delta x^2) \tag{4.3.21}$$

or

$$\frac{C}{4}u_{i-1}^{n+1} - u_i^{n+1} - \frac{C}{4}u_{i+1}^{n+1} = -\frac{C}{4}u_{i-1}^n - u_i^n + \frac{C}{4}u_{i+1}^n \tag{4.3.22}$$

Examples of the numerical solution procedure for a typical first order hyperbolic equation using the explicit and implicit schemes are shown in Section 4.7.3.

4.3.3 MULTISTEP (SPLITTING, PREDICTOR-CORRECTOR) METHODS

Computational stability, convergence, and accuracy may be improved using multistep (intermediate step between n and $n+1$) schemes, such as Richtmyer, Lax-Wendroff, and McCormack methods. The two-step schemes for these methods are shown below.

Richtmyer Multistep Scheme

Step 1

$$\frac{u_i^{n+\frac{1}{2}} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t/2} = -a\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} \tag{4.3.23a}$$

Step 2

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{\left(u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}\right)}{2\Delta x} \tag{4.3.23b}$$

These equations can be rearranged in the form

Step 1

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{C}{4}\left(u_{i+1}^n - u_{i-1}^n\right) \tag{4.3.24a}$$

Step 2

$$u_i^{n+1} = u_i^n - \frac{C}{2}\left(u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.24b}$$

This scheme is stable for $C \leq 2$.

Lax-Wendroff Multistep Scheme

Step 1

$$u_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{i+1}^n + u_i^n\right) - \frac{C}{2}\left(u_{i+1}^n - u_i^n\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.25a}$$

Step 2

$$u_i^{n+1} = u_i^n - C\left(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} - u_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.25b}$$

The stability condition is $C \leq 1$. Note that substitution of (4.3.25a) into (4.3.25b) recovers the original Lax-Wendroff equation (4.3.18). The same result is obtained with (4.3.24a) and (4.3.24b).

MacCormack Multistep Scheme

Here we consider an intermediate step u_i^* which is related to $u_i^{n+\frac{1}{2}}$:

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_i^n + u_i^*\right) \tag{4.3.26}$$

Step 1

$$\frac{u_i^* - u_i^n}{\Delta t} = -a\frac{\left(u_{i+1}^n - u_i^n\right)}{\Delta x} \tag{4.3.27a}$$

Step 2

$$\frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\Delta t/2} = -a\frac{\left(u_i^* - u_{i-1}^*\right)}{\Delta x} \tag{4.3.27b}$$

Substituting (4.3.26) into (4.3.27b) yields

Predictor

$$u_i^* = u_i^n - C\left(u_{i+1}^n - u_i^n\right) \tag{4.3.28a}$$

Corrector

$$u_i^{n+1} = \frac{1}{2}\left[(u_i^n + u_i^*) - C(u_i^* - u_{i-1}^*)\right], \quad O(\Delta t^2, \Delta x^2) \tag{4.3.28b}$$

with the stability criterion of $C \leq 1$.

The MacCormack multistep method is well suited for nonlinear problems. It becomes equivalent to the Lax-Wendroff method for linear problems.

4.3.4 NONLINEAR PROBLEMS

A classical nonlinear first order hyperbolic equation is the Euler's equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \tag{4.3.29}$$

which in conservation form may be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \tag{4.3.30a}$$

or

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x} \quad \text{with } F = \left(\frac{u^2}{2}\right) \tag{4.3.30b}$$

The solution of (4.3.30b) may be obtained by several methods: Lax method, Lax-Wendroff method, MacCormack method, and Beam-Warming implicit method. These are described below.

Lax Method

In this method, the FTCS differencing scheme is used.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x}, \quad O(\Delta t, \Delta x^2) \tag{4.3.31}$$

To maintain stability, we replace u_i^n by its average,

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2\Delta x}(F_{i+1}^n - F_{i-1}^n) \tag{4.3.32}$$

or

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{4\Delta x}\left[(u_{i+1}^n)^2 - (u_{i-1}^n)^2\right] \tag{4.3.33}$$

The solution will be stable if

$$\left|\frac{\Delta t}{\Delta x}u_{\max}\right| \leq 1 \tag{4.3.34}$$

Lax-Wendroff Method

In this method, the finite difference equation is derived from the Taylor series expansion,

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + \cdots \tag{4.3.35}$$

Using (4.3.30b) we have

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right) \tag{4.3.36}$$

where

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial F}{\partial u}\left(-\frac{\partial F}{\partial x}\right) = -A\frac{\partial F}{\partial x} \tag{4.3.37}$$

with A being the Jacobian.

$$A = \frac{\partial F}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u^2}{2}\right) = u \tag{4.3.38}$$

Thus

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\left(-A\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right) \tag{4.3.39}$$

Substituting (4.3.39) and (4.3.30b) into (4.3.35) yields

$$u_i^{n+1} = u_i^n + \left(-\frac{\partial F}{\partial x}\right)\Delta t + \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\frac{\Delta t^2}{2} + O(\Delta t^3)$$

or

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{\partial F}{\partial x} + \frac{\partial}{\partial x}\left(A\frac{\partial F}{\partial x}\right)\frac{\Delta t}{2} + O(\Delta t^2)$$

Approximating the spatial derivatives by central differencing of order 2,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} + \frac{\Delta t}{2\Delta x}\left[\left(A\frac{\partial F}{\partial x}\right)_{i+\frac{1}{2}}^n - \left(A\frac{\partial F}{\partial x}\right)_{i-\frac{1}{2}}^n\right] \tag{4.3.40}$$

The last term above is approximated as

$$\frac{\left(A\frac{\partial F}{\partial x}\right)_{i+\frac{1}{2}}^n - \left(A\frac{\partial F}{\partial x}\right)_{i-\frac{1}{2}}^n}{\Delta x} = \frac{A_{i+\frac{1}{2}}^n \frac{F_{i+1}^n - F_i^n}{\Delta x} - A_{i-\frac{1}{2}}^n \frac{F_i^n - F_{i-1}^n}{\Delta x}}{\Delta x}$$

$$= \frac{\frac{1}{2\Delta x}(A_{i+1}^n + A_i^n)(F_{i+1}^n - F_i^n) - \frac{1}{2\Delta x}(A_i^n + A_{i-1}^n)(F_i^n - F_{i-1}^n)}{\Delta x} \tag{4.3.41}$$

For $A = u$, we obtain

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}(F_{i+1}^n - F_{i-1}^n)$$
$$+ \frac{1}{4}\frac{\Delta t^2}{\Delta x^2}\left[(u_{i+1}^n + u_i^n)(F_{i+1}^n - F_i^n) - (u_i^n + u_{i-1}^n)(F_i^n - F_{i-1}^n)\right] \tag{4.3.42}$$

This is second order accurate with the stability requirement,

$$\left|\frac{\Delta t}{\Delta x}u_{\max}\right| \le 1$$

MacCormack Method

In this method, the multilevel scheme is used as given by

$$u_i^* = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_i^n\right) \tag{4.3.43a}$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + u_i^* - \frac{\Delta t}{\Delta x}(F_i^* - F_{i-1}^*)\right] \tag{4.3.43b}$$

Because of the two-level splitting, the solution performs better than the Lax method or the Lax-Wendroff method. One of the most widely used implicit schemes is the Beam-Warming method, discussed below.

Beam-Warming Implicit Method

Let us consider the Taylor series expansion,

$$u(x, t+\Delta t) = u(x,t) + \left.\frac{\partial u}{\partial t}\right|_{x,t}\Delta t + \left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t}\frac{\Delta t^2}{2} + \mathrm{O}(\Delta t^3) \tag{4.3.44}$$

and

$$u(x,t) = u(x, t+\Delta t) - \left.\frac{\partial u}{\partial t}\right|_{x,t+\Delta t}\Delta t + \left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t+\Delta t}\frac{\Delta t^2}{2!} + \mathrm{O}(\Delta t^3) \tag{4.3.45}$$

Subtracting (4.3.45) from (4.3.44)

$$\begin{aligned} 2u(x,t+\Delta t) &= 2u(x,t) + \left.\frac{\partial u}{\partial t}\right|_{x,t}\Delta t + \left.\frac{\partial u}{\partial t}\right|_{x,t+\Delta t}\Delta t \\ &\quad + \left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t}\frac{\Delta t^2}{2!} - \left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t+\Delta t}\frac{\Delta t^2}{2!} + \mathrm{O}(\Delta t^3) \end{aligned}$$

or

$$u_i^{n+1} = u_i^n + \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^n + \left(\frac{\partial u}{\partial t}\right)_i^{n+1}\right]\Delta t + \frac{1}{2}\left[\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \left(\frac{\partial^2 u}{\partial t^2}\right)_i^{n+1}\right]\frac{\Delta t^2}{2!} + \mathrm{O}(\Delta t^3)$$

where

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_i^{n+1} = \left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n \Delta t + \mathrm{O}(\Delta t^2)$$

Thus, we arrive at

$$u_i^{n+1} = u_i^n + \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^n + \left(\frac{\partial u}{\partial t}\right)_i^{n+1}\right]\Delta t + \mathrm{O}(\Delta t^3) \tag{4.3.46}$$

For the model equation

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x} \tag{4.3.47}$$

Using (4.3.46) in (4.3.47), we obtain

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left[\left(\frac{\partial F}{\partial x}\right)_i^n + \left(\frac{\partial F}{\partial x}\right)_i^{n+1}\right] + \mathrm{O}(\Delta t^2) \tag{4.3.48}$$

This indicates that (4.3.48) leads to the second order accuracy.

Recall that the nonlinear term $F = u^2/2$ was applied at the known time level n, and the resulting FDE in explicit form was linear. The resulting FDE in implicit formulation is nonlinear, and therefore a procedure is used to linearize the FDE. To this end, we write a Taylor series for $F(t + \Delta t)$ in the form

$$\begin{aligned} F(t+\Delta t) &= F(t) + \frac{\partial F}{\partial t}\Delta t + O(\Delta t^2) \\ &= F(t) + \frac{\partial F}{\partial u}\frac{\partial u}{\partial t}\Delta t + O(\Delta t^2) \end{aligned}$$

or

$$F^{n+1} = F^n + \frac{\partial F}{\partial u}\left(\frac{u^{n+1} - u^n}{\Delta t}\right)\Delta t + O(\Delta t^2) \tag{4.3.49}$$

Taking a partial derivative of (4.3.49) yields

$$\left(\frac{\partial F}{\partial x}\right)^{n+1} = \left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}[A(u^{n+1} - u^n)] \tag{4.3.50}$$

Combining (4.3.48) and (4.3.50) gives

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left\{\left(\frac{\partial F}{\partial x}\right)_i^n + \left(\frac{\partial F}{\partial x}\right)_i^n + \frac{\partial}{\partial x}[A(u_i^{n+1} - u_i^n)]\right\}$$

or

$$u_i^{n+1} = u_i^n - \frac{1}{2}\Delta t\left\{2\left(\frac{\partial F}{\partial x}\right)_i^n + \frac{\partial}{\partial x}[A(u_i^{n+1} - u_i^n)]\right\} \tag{4.3.51}$$

Using a second order central differencing for the terms with A on the right-hand side of (4.3.51) and linearizing, we obtain

$$\begin{aligned} u_i^{n+1} = u_i^n - \frac{1}{2}\Delta t\Bigg[&\frac{2(F_{i+1}^n - F_{i-1}^n)}{2\Delta x} + \frac{A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}}{2\Delta x} \\ &- \frac{A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n}{2\Delta x}\Bigg] \end{aligned} \tag{4.3.52}$$

Modifying (4.3.52) to a tridiagonal form

$$\begin{aligned} &-\frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^{n+1} \\ &= u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}(F_{i+1}^n - F_{i-1}^n) + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^n - \frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^n + D \end{aligned} \tag{4.3.53}$$

This scheme is second order accurate, unconditionally stable, but dispersion errors may arise. To prevent this, a fourth order smoothing (damping) term is explicitly added:

$$D = -\frac{\omega}{8}(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n),$$

with $0 < \omega < 1$. Since the added damping term is of fourth order, it does not affect the second order accuracy of the method.

4.3.5 SECOND ORDER ONE-DIMENSIONAL WAVE EQUATIONS

Let us consider the second order one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{4.3.54}$$

Here we require two sets of initial conditions,

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

and two sets of boundary conditions,

$$u(0,t) = h_1(t)$$
$$u(L,t) = h_2(t)$$

We may use the midpoint leapfrog method for this problem,

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + C^2\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) \tag{4.3.55}$$

If we choose $\dfrac{\partial u(x,0)}{\partial t} = 0$, then

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = 0$$

or

$$u_i^{n+1} = u_i^{n-1}$$

Thus, from (4.3.55), we obtain

$$u_i^{n+1} = u_i^n + \frac{1}{2}C^2\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) \tag{4.3.56}$$

This is called the midpoint leapfrog method. An example problem for the second order hyperbolic equation is demonstrated in Section 4.7.4.

4.4 BURGERS' EQUATION

The Burgers' equation is a special form of the momentum equation for irrotational, incompressible flows in which pressure gradients are neglected. It is informative to study this equation in the one-dimensional case before we launch upon full-scale CFD problems.

Consider the Burgers' equation written in various forms:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4.4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4.4.2}$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4.4.3}$$